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A (d,c,v)-graph is a c-connected graph of diameter = d in which each node is of valence = v. The minimum order (number of nodes) of such graphs is denoted by micron(d,c,v), and a minimum (d,c,v)-graph is one of minimum order. Each minimum (d,c,v)-graph corresponds to an efficient way of arranging the stations of a communication network so that if any c - 1 stations are incapacitated, the rest of the network is still connected, and so that in case of breakdown or other difficulty, each station can rely for assistance on precisely v others. The present paper classifies and counts the minimum (d,1,3)-graphs and the minimum (d,2,3)-graphs, a task performed elsewhere for the minimum (d,3,3)-graphs.

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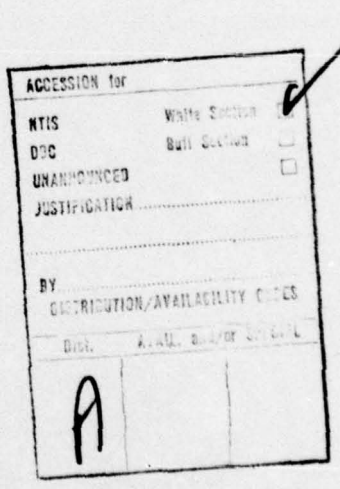

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Classification and Enumeration of Minimum (d, 1, 3)-Graphs and Minimum (d, 2, 3)-Graphs

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A (d,c,v)-graph is a c-connected graph of diameter = d in which each node is of valence = v. The minimum order (number of nodes) of such graphs is denoted by $\mu(d,c,v)$, and a minimum (d,c,v)-graph is one of minimum order. Each minimum (d,c,v)-graph corresponds to an efficient way of arranging the stations of a communication network so that if any c - 1 stations are incapacitated, the rest of the network is still connected, and so that in case of breakdown or other difficulty, each station can rely for assistance on precisely v others (see [2]).

The $\langle d,c,v\rangle$-graphs and the function $\mu\langle d,c,v\rangle$ are defined by replacing = with $\geq$ in the above definitions. The functions $\mu\langle d,c,v\rangle$ and $\mu(d,c,v)$ are determined in [2] and [3] respectively. The present paper classifies and counts the minimum (d,1,3)-graphs and the minimum (d,2,3)-graphs, a task performed in [1] for the minimum (d,3,3)-graphs.

THE CASE $d \leq 4$

The following can be established by a routine but tedious division into cases. Details are left to the reader.

PROPOSITION For $1 \leq d \leq 4$ the minimum $(d, 1, 3)$-graphs and the minimum $(d, 2, 3)$-graphs are precisely those shown below.

(Figure 1 is to be inserted here)

Minimum $(d,1,3)$-graphs and minimum $(d,2,3)$-graphs for $d \leq 4$

It is assumed henceforth that $d \geq 5$, except where a different inequality is explicitly stated.

DIAMOND STRINGS AND MINIMUM (d, 1, 3)-GRAPHS

A clasp and a diamond are the graphs shown below, where each pendant edge is later to be combined with another such edge in forming an ordinary undirected graph.

(Figure 2 is to be inserted here)

Clasp Diamond

The diamond string D(k) with k diamonds is formed by combining two clasps and k diamonds in the manner shown below.

(Figure 3 is to be inserted here)

D(0) D(1)

(Figure 4 is to be inserted here)

clasp k diamonds clasp

The diamond string D(k) with k diamonds

An enlarged clasp is one of the two graphs

(Figure 5 is to be inserted here)

Enlarged clasps

and a doubly enlarged clasp is one of the three graphs

(Figure 6 is to be inserted here)

Doubly enlarged clasps

An enlarged diamond is one of the two graphs

(Figure 7 is to be inserted here)

Enlarged diamonds

and a doubly enlarged diamond is one of the two graphs

(Figure 8 is to be inserted here)

Doubly enlarged diamonds

THEOREM For integers $5 \le j \le 7$ and $k \ge 0$, $\mu(j + 3k,1,3) = 2j + 4k$. The unique minimum $(5 + 3k, 1, 3)$-graph is the diamond string $D(k)$ with k diamonds. The minimum $(6 + 3k, 1, 3)$-graphs are the graphs formed from $D(k)$ by enlarging one clasp or one diamond; their number is $2 + k$ for even k and $3 + k$ for odd k. The minimum $(7 + 3k, 1, 3)$-graphs are the graphs formed from $D(k)$ by doubly enlarging one clasp, enlarging both clasps, enlarging one clasp and one diamond, enlarging two diamonds, or doubly enlarging one diamond; their number is $6 + 5k + [3k^2/2]$.

Proof. Plainly

(1) $\mu(d,c,v) \ge \mu\langle d,c,v \rangle$, with strict inequality when both d and $\mu\langle d,c,v \rangle$ are odd,

because odd-valent graphs are of even order. That

(2) $\mu(j + 3k, 1, 3) \geq 2j + 4k$

follows from (1) and the fact [2] [4] that $\mu\langle d,1,3\rangle = 4 + d + [d/3]$. It is easily verified that $D(k)$ and the graphs derived from it have the indicated diameters and orders. Since they are plainly connected and 3-valent, equality holds in (2). It remains to show that for $d \geq 5$ there are no other minimum $(d,1,3)$-graphs and, having done that, to justify the formulas for the number of such graphs.

With $5 \leq j \leq 7$, $k \geq 0$, and $d = j + 3k$, let G be a minimum $(d,1,3)$-graph, let $P = (x_o,x_1,\ldots,x_d)$ be a path of length d joining the two nodes x_o and x_d of a diametral pair, and let v and w [resp. y and z] be the other two nodes of G adjacent to x_o [resp. x_d]. Let

$V = \{v,w,x_o,x_1,x_2\}$, $X = \{x_i: 3 \leq i \leq d - 3\}$, $Z = \{x_{d-2},x_{d-1},x_d,y,z\}$

and let T [resp. Q] be the set of all nodes of G that have not yet been named and have 3 [resp. < 3] neighbors in X. Let

$$m = |Q \cup T| = \mu(d,1,3) - (d + 5) = j + k - 5$$

and

$$e = |X| = j + 3k - 5.$$

Note that each node of X is incident to a unique edge that is not on the path P, and since P is a shortest path from x_o to x_d the edge in question always has its other end in $Q \cup T$. From the shortness of P it follows also that

(3) $|h - i| \leq 2$ whenever x_h and x_i have a common neighbor, and

(4) the neighbors of a node in T are three consecutive nodes in X.

If $j = 5$ then $m = k$ and $e - 3k$, whence Q is empty and it follows from (4) that G has the spanning subgraph shown in the next figure. But then plainly G is $D(k)$.

(Figure 9 is to be inserted here)

Spanning subgraph of minimum (5+3k,1,3)-graph

We still must consider the two cases:

(5) $j = 6, \quad m = 1 + k, \quad e = 1 + 3k;$

(6) $j = 7, \quad m = 2 + k, \quad e = 2 + 2k.$

Define the multiplicity of a node of Q as the number of edges joining it to X, and note that the total number of edges joining Q to X is

$$(7) \quad e - 3|T| = e - 3(m - |Q|) = \begin{cases} 3|Q| - 2 & \text{when (5) holds.} \\ 3|Q| - 4 & \text{when (6) holds.} \end{cases}$$

Note also the following consequences of P's shortness:

(8) no edge joins V to Z;

(9) if a node of G has a neighbor in V and also a neighbor in Z then $d = j = 6$, $k = 0$, and the neighbors are x_2 and x_4 respectively.

Now suppose (5) holds, whence by (7) the sequence of multiplicities of the members of Q is (1) or (2,2). In the first instance Q consists of a single node q, q has a single neighbor in X, and q's other two neighbors belong to $V \cup Z$. From (4) in conjunction with P's shortness it follows that q's neighbor in X is x_3 or x_{d-3}. We may assume it is the former, whence G has the spanning subgraph shown in the next figure.

(Figure 10 is to be inserted here)

Spanning subgraph of minimum (6+3k,1,3)-graph when $Q = \{q\}$

It is readily verified that x_1 and x_2 have no common neighbor, and since v and w are interchangeable we may assume one of the following holds: x_1 is adjacent to v and x_2 to q; x_1 is adjacent to q and x_2 to w; x_1 is adjacent to v and x_2 to w. The first case yields the first of the enlarged clasps described earlier, and the other two cases both yield the second of the enlarged clasps.

Now suppose (5) holds and Q's sequence of multiplicities is (2,2). If a node u of $V \cup Z$ is adjacent to Q it follows from (3) in conjunction with P's shortness that u is x_2 or x_{d-2}, which is quickly seen to be impossible. Hence the two nodes of Q are neighbors, whence, calling again on (3) and on P's shortness, we see that for some i with $3 \le i < i + 3 \le 3 + 3k$, G has one of the two subgraphs shown in the next figure.

(Figure 11 is to be inserted here)

Possible subgraphs of minimum (6+3k,1,3)-graph when $|Q| = 2$

It follows with the aid of (6) that i is a multiple of 3 and hence the above subgraphs correspond to the situation in which G is obtained from a diamond chain by enlarging one diamond.

It was proved in the ceding two paragraphs that the minimum (6+3k,1,3)-graphs are as claimed. Their number is also as claimed,

for there are (for a given $k \geq 0$) two different isomorphism types having an enlarged clasp and $2\lceil k/2 \rceil$ types having an enlarged diamond, where $\lceil k/2 \rceil$ is the smallest integer $\geq k/2$. Only the case $j = 7$ remains.

If $j = 7$ — that is, if (6) holds — Q's sequence of multiplicities is limited by (7) to the following possibilities:

$$(2,0), \quad (1,1), \quad (2,2,1), \quad (2,2,2,2).$$

The analysis of these possibilities is similar to (though more complicated than) the analysis provided above for the case $j = 6$. Details are omitted, but the conclusion is that the minimum $(7+3k,1,3)$-graphs are as claimed in the theorem. To see that their number is as claimed, note that, in view of the numbers of the various types of enlargements and the asymmetry of one of the doubly enlarged diamonds, the number of minimum $(7+3k,1,3)$-graphs having

a doubly enlarged clasp is 3;

two enlarged clasps is 3;

an enlarged clasp and an enlarged diamond is $4k$;

a doubly enlarged diamond is $\lceil k/2 \rceil + \cdots + k$;

two enlarged diamonds of different types is

$$(k-1) + \cdots + (k-2) + \cdots + 1 = k(k-1)/2;$$

two enlarged diamonds of the same type is $2s$, where

$$k \text{ even} \Rightarrow s = (k-1) + (k-3) + \cdots + 1 = \frac{k^2}{2} = \left[\frac{k^2}{2}\right]$$

and

$$k \text{ odd} \Rightarrow s = (k-1) + (k-3) + \cdots + 2 = \frac{k^2-1}{2} = \left[\frac{k^2}{2}\right].$$

Hence the total number is

$$6 + \tfrac{9}{2}k + \tfrac{1}{2}k^2 + \lceil k/2 \rceil + 2[k^2/2] = 6 + 5k + [3k^2/2].$$

DEMON LADDERS AND MINIMUM (d, 2, 3)-GRAPHS

For each positive integer k, a k-ladder is formed from two node-disjoint simple paths $(u_1,\ldots,u_k)$ and $(v_1,\ldots,v_k)$ by adding k additional edges (the rungs) which match the u_i's with the v_j's in such a way that $|i - j| \leq 1$ whenever u_i is matched with v_j, and also adding pendant edges at u_1, u_k, v_1 and v_k. (When $k = 1$ there are two pendant edges at u_1 and two at v_1.) For example, each 3-ladder is ismorphic to one of the following.

(Figure 12 is to be inserted here)

The two 3-ladders

A small end, a large end, and a forked end are shown below, each having two pendant edges. Note that each large end contains a small end.

(Figure 13 is to be inserted here)

Small end Large end Forked end

A demon ladder with k rungs in formed by placing a k-ladder between two small ends in the manner shown below for $k = 2$. Note that a demon ladder may have a large end but is not required to.

(Figure 14 is to be inserted here)

Demon ladder with two rungs

As follows from the theorem below, the minimum (5,2,3)-graphs are precisely the above two demon ladders and the two graphs shown below.

(Figure 15 is to be inserted here)

Other minimum (5,2,3)-graphs

THEOREM *For all* d, $\mu(d,2,3) = 2d + 2$. *For* $d \geq 4$ *the minimum* $(d,2,3)$*-graphs are the demon ladders with* $d - 3$ *rungs and the graphs obtained from such demon ladders by replacing a large end with a forked end or two large ends with forked ends. For* $d \geq 5$ *the number of minimum* $(d,2,3)$*-graphs is*

$$\tfrac{1}{2}f(d - 1) + \tfrac{1}{2}f\left(\frac{2d - 1 - (-1)^d 3}{4}\right)$$

where $f(k)$ *is the* k*th Fibonacci number.*

Proof. It is easily verified that the demon ladders and their derivatives are $(d,2,3)$-graphs, whence

(10) $$\mu(d,2,3) \leq 2d + 2.$$

Now suppose that a G is minimum $(d,2,3)$-graph with $d \geq 4$, let $\{x,y\}$ be a diametral pair of nodes, and let $P = (x,p_1,p_2,\ldots,y)$ and $Q = (x,q_1,q_2,\ldots,y)$ be a pair of independent paths from x to y such that, among all such pairs, the sum of the lengths of P and Q is a minimum. Plainly x [resp. y] has a neighbor

w [resp. z] not in PuQ. Since each of P and Q has at least d - 1 intermediate nodes, and since $w \neq z$, it follows with the aid of (10) that $\mu(d,2,3) = 2d + 2$, P and Q are both of length d, and w and z are the only nodes of G not in PuQ. Thus G has the spanning subgraph shown in the next figure and it remains only to consider the possibilities for the remaining edges.

(Figure 16 is to be inserted here)

Spanning subgraph of minimum (d,2,3)-graph

Plainly w's two neighbors other than x belong to $\{p_1,p_2,q_1,q_2\}$. If w is adjacent to both p_1 and p_2 [resp. q_1 and q_2] then all possibilities for the third neighbor of q_1[resp. p_1] lead to contradictions and hence w's set of neighbors is $\{x,p_1,q_1\}$, $\{x,p_1,q_2\}$, $\{x,p_2,q_1\}$ or $\{x,p_2,q_2\}$. In the first of these cases, w belongs to a small end, and to a large end if $\{p_2,q_2\}$ is an edge. The last three cases imply respectively the adjacency of p_2 to q_1, of p_1 to q_2, and of p_1 to q_1, and hence lead to forked ends in the manner shown below.

(Figure 17 is to be inserted here)

Three ways of obtaining a forked end

Similar considerations apply to z's neighbors. Since the shortness of P and Q implies $|i - j| \leq 1$ whenever p_i or

q_i is adjacent to p_j or q_j, it is now clear that the minimum $(d,2,3)$-graphs are precisely as described in the Theorem. It remains only to count the number of isomorphism types of such graphs.

Let $f(k)$ denote the number of ordered partitions of k into 1's and 2's — that is, the number of sequences $(a_1,\ldots,a_b)$ such that $a_i \in \{1,2\}$ for all i and $\Sigma_1^b a_i = k$. To see that $f(k)$ is the kth Fibonacci number, note that $f(1) = 1$, $f(2) = 2$, and

(11) $$f(k) = f(k - 1) + f(k - 2)$$

because $f(k-1)$ [resp. $f(k-2)$] is the number of sequences $(a_1,\ldots,a_b)$ of 1's and 2's such that $\Sigma_1^b a_i = k$ and $a_1 = 1$ [resp. $a_1 = 2$]. Let $s(k)$ denote the number of partitions, among those counted by $f(k)$, that are symmetric (equal to their own reverses) and note that

(12) when k is odd, $s(k) = f(\frac{k - 1}{2})$;

(13) When k is even $s(k) = f(\frac{k}{2}) + f(\frac{k}{2} - 1) = f(\frac{f}{2} + 1)$.

For each ordered partition $a = (a_1,\ldots,a_b)$ of k into 1's and 2's, let L_a denote the k-ladder formed from two node-disjoint simple paths by dividing the nodes into b blocks — the first block consisting of the first a_1 u_i's together with the first a_1 v_i's, the second block consisting of the next a_2 u_i's together with the next a_2 v_i's, etc. — and then adding edges (in addition to the four pendant edges) according to the following rules:

if $\{u_i,v_i\}$ is a block it is also an edge;

if $\{u_i,u_{i+1},v_i,v_{i+1}\}$ is a block then $\{u_i,v_{i+1}\}$ and $\{u_{i+1},v_i\}$ are edges.

An example is shown below.

(Figure 18 is to be inserted here)

The 8-ladder $L_{(1,2,2,1,2)}$

Note that

(14) two k-ladders L_a and $L_{\bar{a}}$ are isomorphic if and only if the sequences a and $\bar{a}$ are equal or one is the reverse of the other.

With the aid of (11)-(14) we can count the number of isomorphism types of minimum (d,2,3)-graphs. Let $r = d - 3$. Then it is not hard to verify that there are

$$s(r) + \tfrac{1}{2}(f(r) - s(r)) = \tfrac{1}{2}s(r) + \tfrac{1}{2}f(r)$$

types of minimum (d,2,3)-graphs with two small ends,

$$f(r - 1)$$

types with one small end and one forked end, and

$$\tfrac{1}{2}s(r - 2) + \tfrac{1}{2}f(r - 2)$$

types with two forked ends. Hence the total number of types is the sum of

$$t = \tfrac{1}{2}s(r) + \tfrac{1}{2}s(r - 2)$$

and

$$\tfrac{1}{2}(f(r)+f(r-1)) + \tfrac{1}{2}(f(r-1) + f(r-2)) = \tfrac{1}{2}f(r+1) + \tfrac{1}{2}f(r) = \tfrac{1}{2}f(r+2) = \tfrac{1}{2}f(d-1$$

When r is even it follows from (12) that

$$2t = f(\tfrac{r}{2}+1) + f(\tfrac{r-2}{2} + 1) = f(\tfrac{r}{2}+2) = f(\tfrac{d+1}{2}) = f(\tfrac{2d-1-(-1)^{d}3}{4})$$

and when r is odd it follows from (13) that

$$2t = f(\tfrac{r-1}{2}) + f(\tfrac{r-3}{2}) = f(\tfrac{r+1}{2}) = f(\tfrac{d-2}{2}) = f(\tfrac{2d-1-(-1)^{d}3}{4})$$

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d=1
d=2
d=3
d=4

(4)
(3)
(2)

(7)

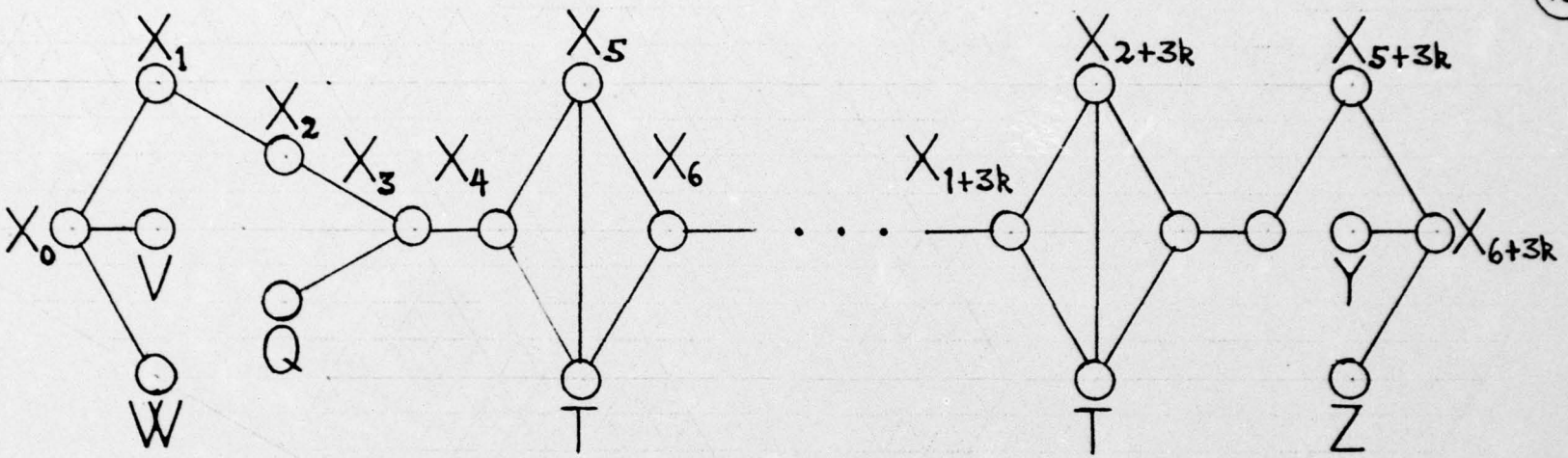
(6)

(5)

(9)

(8)

(10)


X1
X2
X3
X4
X5
X6
X0
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X1+3k
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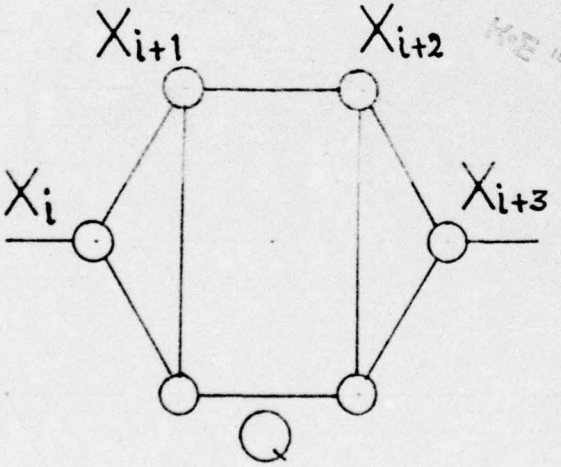
X6+3k
Y
Z
T

(11)


Xi+1
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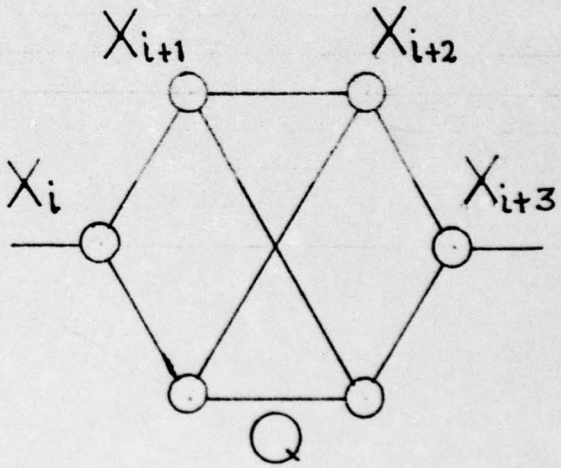
Xi
Xi+3
Q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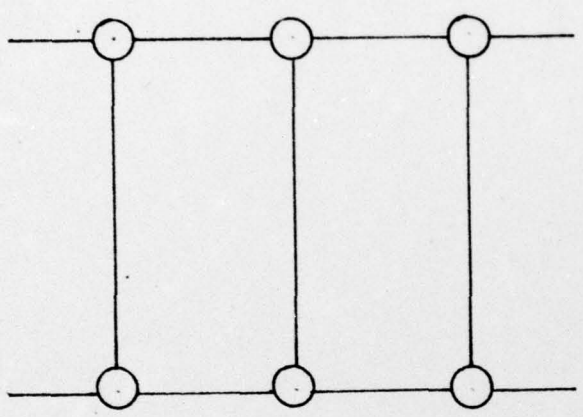
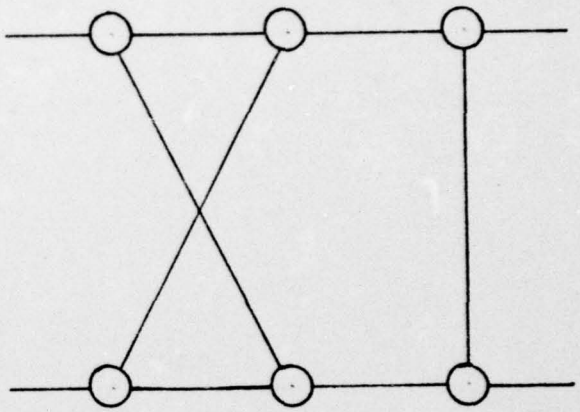
Q

(12)





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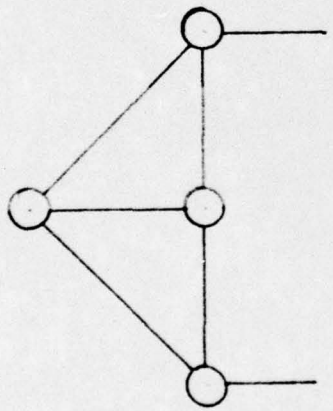

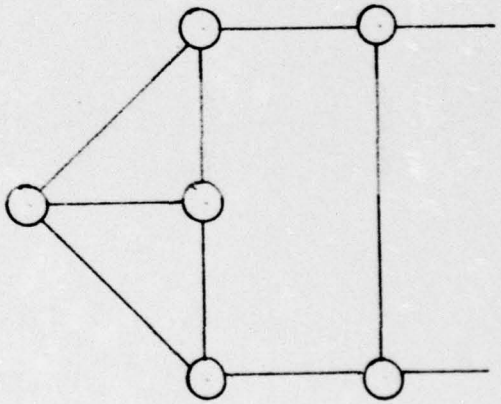

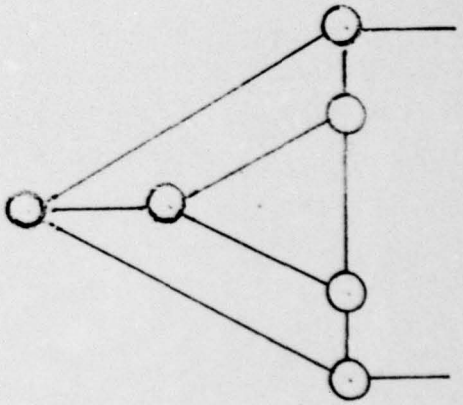

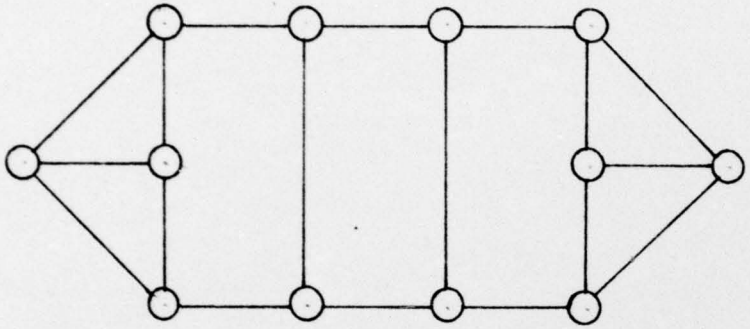

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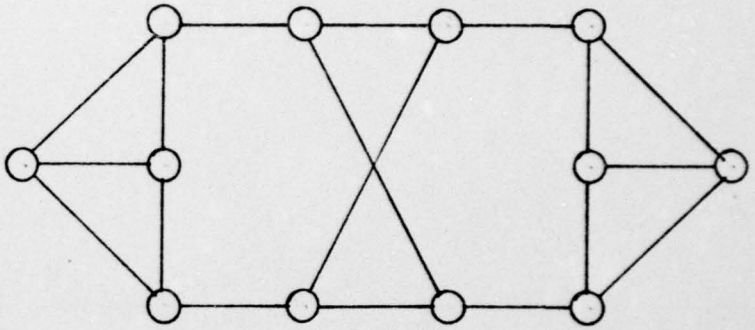

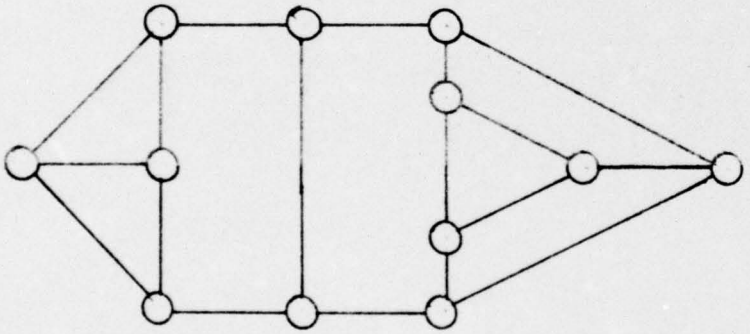

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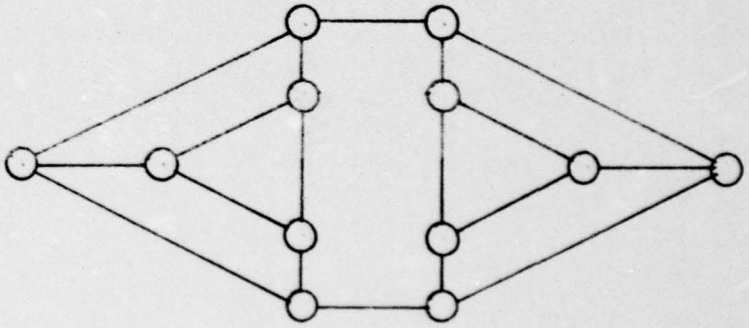

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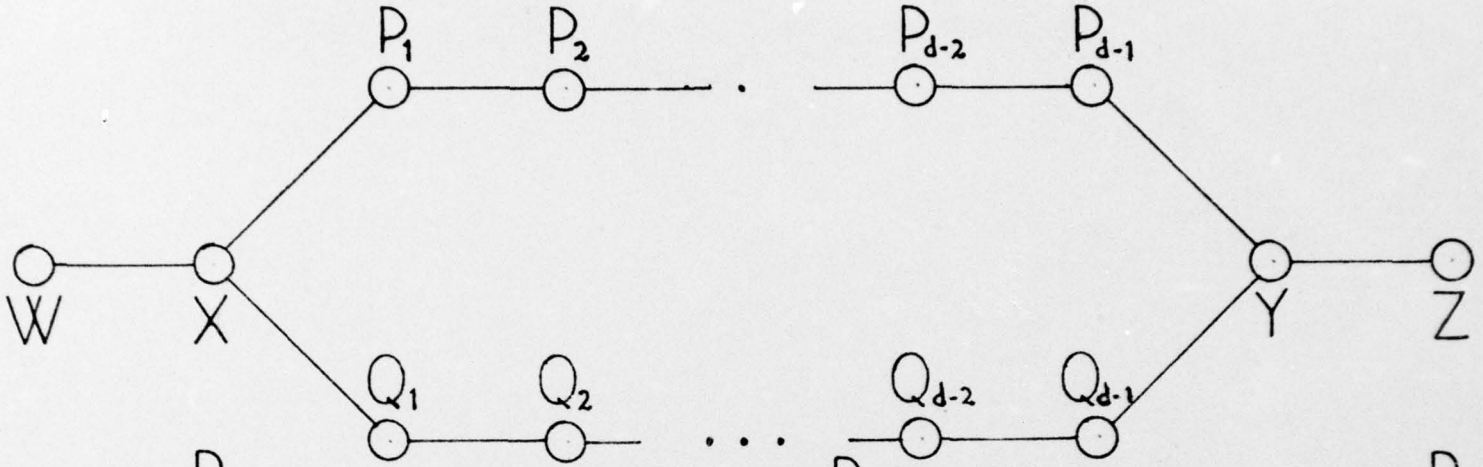
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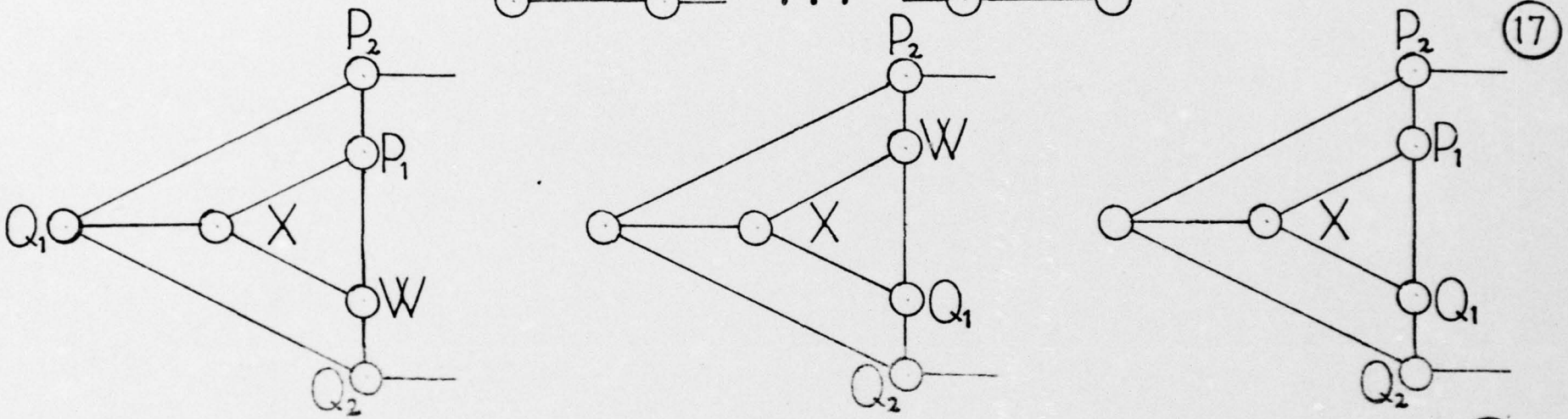
Pd-1
W
X
Y
Z
Q1
Q2
Qd-2
Qd-1

(17)


P2
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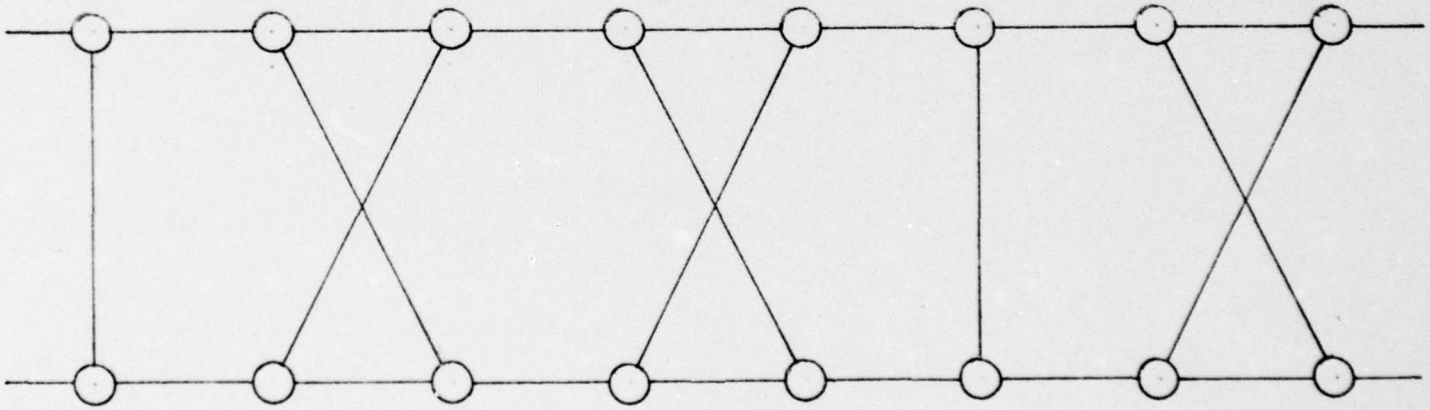
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